A NOTE ON FOUR AND SIX LEVEL SECOND ORDER ROTATABLE DESIGNS

By

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1. Introduction

Balanced Incomplete Block Designs (BIBD) have been used by many workers to construct rotatable designs. Nigam and Dey [5] were the first to obtain four and six level second order rotatable designs (SORD) through factorial combinations of the levels. This paper aims at suggesting the use of BIBD in constructing four and six level SORD.

2. Four Level SORD

Let $N(=n_{ij})$ be the $b \times v$ incidence matrix of a BIBD with parameters (v, k, r, b, λ) such that $n_{ij}=\alpha$ if the j^{th} treatment occurs in i^{th} block and $n_{ij}=\beta$ otherwise.

Evidently N is a $b \times v$ array of symbols (α, β). Then, by following Das and Narasimham [1], these will generate $b2^p$ combinations of symbols α and β on multiplying the b rows with combinations of 2^p factorial where the levels of the 2^p factorial are 1 and -1 and 2^p is the number of combinations in a resolution V fraction of 2^V. It can easily be verified that these $b2^p$. level combinations of α and β form a rotatable arrangement in v dimensions if $r \neq 3\lambda$ (*See* also, Dey and Kulshreshtha [2]). From these, four level second order rotatable designs can easily be obtained by omitting any x columns of the rotatable arrangement. This result was also independently obtained by Saha and Das [6].

Alternatively, four level SORD can be obtained from the $b2^p$. combinations of the levels $\pm\alpha$ and $\pm\beta$ obtained above by adding to theses suitable level combinations of $\pm\alpha$ and $\pm\beta$. We take up the cases when (*i*) $r \geqslant 3\lambda$ and (*ii*) $r < 3\lambda$, separately.

Case 1. $r \geqslant 3\lambda$

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In this case four level SORD can be constructed by adding to the $b2^p$ initial level combinations, a set of $v.2^{p'}$ level combinations obtained through permutations of $(\pm\alpha\pm\beta\ldots\pm\beta)$. It may be seen that these SORD will have $(b+v)2^p$ design points. The solution of the condition

$$\sum x_{ju}^4=3\sum x_{iu}^2\, x_{ju}^2$$

comes out to be

$$\begin{aligned} r\alpha^4+(b-r)\,\beta^4+\alpha^4+(v-1)\,\beta^4 \\ =3[\alpha^4\lambda+(b-2r+\lambda)\,\beta^4+2(r-\lambda)\alpha^2\beta^2] \\ +3[2\alpha^2\beta^2+(v-2)\,\beta^4], \end{aligned}$$

which on simplifications gives

$$[(r-3\lambda)+1]\,t^2-6\,[(r-\lambda)+1]\,t-(3\lambda+2b-5r)-(2v-5)=0,$$

where

$$t=\alpha^2/\beta^2.$$

It can easily be shown that this equation gives a positive solution of t when $r\geqslant 3\lambda$. We thus obtain four level SORD with $(b+v)2^p$ points.

Case 2. $r<3\lambda$

Four level SORD can be constructed by adding 2^p level combinations obtained through the set $(\pm\alpha\ \pm\alpha\ ..\pm\alpha)$ and $b2^p$ initial level combinations obtained through the BIBD. The solution of

$$\sum x_{iu}^4=3\sum x_{ju}^2\, x_{ju}^2$$

gives the equation

$$\begin{aligned} r\alpha^4+(b-r)\,\beta^4+\alpha^4=3\,[\alpha^4\lambda+(b-2r+\lambda)\,\beta^4 \\ +2(r-\lambda)\,\alpha^2\beta^2]+3\alpha^4 \end{aligned}$$

which on simplification reduces to the form

$$(r-3\lambda-2)t^2-6(r-\lambda)t+(5r-2b-3\lambda)=0.$$

It can easily be seen that this equation yields a positive solution of t when $r<3\lambda$. We thus obtain four level SORD with $(b+1)2^p$ design points.

3. Six Level SORD

Six level SORD can be obtained by taking $b2^p{}_j$ initial level combinations with $\pm\alpha$ and $\pm\beta$ and $v.2^p$ level combinations obtained through v permutations of $(\pm\theta\pm\beta\pm\beta\ldots\pm\beta)$. The solution of

$$\sum x_{iu}^4=3\sum x_{iu}^2\, x_{ju}^2$$

gives the equation

$$r\alpha^4+(b-r)\beta^4+\theta^4+(v-1)\beta^4$$
$$=3[\alpha^4\lambda+(b-2r+\lambda)\beta^4+2(r-\lambda)\alpha^2\beta^2]$$
$$+3[2\theta^2\beta^2+(v-2)\beta^4]$$

which on simplification gives

$$(r-3\lambda)s^2+t^2-6s(r-\lambda)-6t+(5r-2b-2v-3\lambda+5)=0$$

where $\quad s=\alpha^2/\beta^2,\ t=\theta^2/\beta^2.$

If we choose an arbitrary value of s, the above equation reduces to a quadratic equation in t. It can easily be seen that positive solution of t can always be obtained for all the cases $r \geqslant 3\lambda$ and $r<3\lambda$. We therefore get a six level SORD in v factors with $(b+v)2^p$ design points.

Six level SORD can be obtained with only $(b+1)2^p$ design points when $r=3\lambda$ by taking the $b2^p$ initial level combination and 2^p level combinations of the set $(\pm\theta\pm\theta..\pm\theta)$. The solution of

$$\sum x_{iu}^4=3\sum x_{iu}^2\,x_{ju}^2 \text{ gives}$$

$$(b-r)\beta^4+r\alpha^4+\theta^4=3[\alpha^4\lambda+(b-2r+\lambda)\beta^4$$
$$+2(r-\lambda)\alpha^2\beta^2]+3\theta^4$$

which reduces to the form

$$(r-3\lambda)s^2-2t^2-6(r-\lambda)s+(5r-2b-3\lambda)=0.$$

It can easily be seen that by fixing s arbitrarily, a positive solution of t exists when $r=3\lambda$.

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A NOTE ON THE CONSTRUCTION OF PARTIALLY BALANCED TERNARY DESIGNS

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INTRODUCTION

Balanced n-ary block designs were introduced by Tocher (1952). Murthy and Das (1967), Dey (1970), Saha and Dey (1973) and Nigam (1975) gave methods of constructing these designs, in particular balanced ternary designs. Later Mehta, Agarwal and Nigam (1975) introduced partially balanced ternary (PBT) designs as generalisation of PBIB design. A PBT design is defined as follows (Mehta, et al., 1975).

A block design with V treatments and B blocks is said to be a PBT design with p associate classes and parameters

$$V,\ B, R, K,\ \pi_i,\ \triangle,\ n_i, p^i_{jk}\ ,\ i, j, k=1, 2, \cdot\ , p, \text{ if}$$

(*i*) The incidence matrix $N_{B\times V}$ has three entries 0, 1 and 2.

(*ii*) The row sum of N is K.

(*iii*) The column sum of N is R and the column sum of squares is $\triangle$.

(*iv*) There exists a relationship between the treatments defined as—

(*a*) any two treatments are either 1st, 2nd..., or p-th associates, the relation of association being symmetrical;

(*b*) each treatment θ has n_i i-th associates;

(*c*) if any two treatments θ and ϕ are mutually i-th associates, then the number of treatments that are j-th associates of θ and k-th associates of ϕ, is p^i_{jk} and is independent of the pair of i-th associates θ and ϕ;

(v) The inner product of columns θ and ϕ of N is π_i, if θ and ϕ are mutually i-th associates, $i=1,2,\ldots, p$.

The purpose of this note is to give a method of construction of 2-associate cyclic PBT designs using the method of differences.

Method of differences for the construction of cyclic PBT designs

Consider a module M of v elements $0,1,\ldots, v-1$. Let the non-zero elements of M be divided into two disjoint sets D and E of n_1 and n_2 elements respectively, *i.e.*,

$$D=(d_1,d_2\ldots, d_{n_1})$$

and

$$E=(e_1, e_2, \ldots, e_{n_2}).$$

Further, let the set D be such that

$$D=(-d_1,-d_2, \ldots,-d_{n_1})$$

and that among the $n_1(n_1-1)$ differences d_j-d_j' arising out of it, the elements of D are repeated p_{11}^1 times and those of E, p_{11}^2 times.

With each element of M let us associate one treatment. Let the 1st associates of any treatment i be

$$(i+d_1, i+d_2, \ldots, i+d_{n_1})$$

mod(v), and its 2nd associates be

$$(i+e_1, i+e_2,\ldots, i+e_{n_2})$$

mod(v), then the v treatments are said to satisfy a two associate cyclic association scheme with parameters

$$n_i, p_{jk}^i \ ; \ i, j, k=1, 2$$

[cf. Bose and Shimamoto (1952)]

Let it be possible to find t initial blocks $B_1, B_2,\ldots, B_t$ such that:

(i) Every block contains k treatments.

(ii) The treatment i occurs n_{ij} times in B_j, where $n_{ij}=0, 1$, or 2 ($i=0, 1, 2,\ldots, v-1$; $j=1, 2,\ldots, t$). All the treatments do not occur in any of the blocks B_j. There is atleast one pair (i, j) for which $n_{ij}=1$ and atleast one pair (i', j') for which $n_{i'j'}=2$ ($i, i'=0, 1,\ldots, v-1$; $j, j'=1, 2, \ldots, t$), where any of the three possibilities (a) $i=i', j\neq j'$, (b) $i\neq i', j=j'$, (c) $i\neq i', j\neq j'$, may occur.

(iii) The non-zero differences mod(v) from the t blocks consist of all the non-zero treatments

$$d_1, d_2,\ldots, d_{n_1} ; e_1,e_2,\ldots, e_{n_2},$$

with each d occurring π_1 times and each e occurring π_2 times among them.

If i is any treatment, from each B_j, we can form another block $B_{j,i}$, by adding i to the elements of B_j and reducing them mod(v). We then have the following :

Theorem. The vt blocks $B_{j,i}$ ($j=1, 2,\ldots, t$; $i=0, 1,\ldots, v-1$) as constructed above form a cyclic PBT design with parameters $V=v$, $B=vt$, $K=k$, $R=kt$, π_1, π_2, $\triangle=k^2t-\pi_1n_1-\pi_2n_2$,

$$n_i,\ p^i_{jk}\ ;\ i, j, k=1, 2.$$

Proof. The proof of the theorem follows on the usual lines laid down for such results on the method of differences. The value of $\triangle$ is same as the number of zero differences out of k^2t differences arising out of the t initial blocks and is equal to $k^2t-\pi_1n_1-\pi_2n_2$.

Example. Consider a module M of residue classes mod(5). Let $d_1=1$ and $d_2=4$; $e_1=2$, $e_2=3$. It is readily seen that the five treatments 0, 1,2,3,4 satisfy a cyclic association scheme with parameters

$$V=5,\ n_1=n_2=2,\ p^1_{11}=0,\ p^2_{11}=1.$$

Consider the initial blocks (1,1,2) and (1,3,4). The non-zero differences from these blocks are 1,4, 1,4, 2,3,3,2,1,4 so that $\pi_1=3$ and $\pi_2=2$. Developing the initial blocks, we obtain the blocks (1,1,2), (2,2,3), (3,3,4), (4,4,0), (0,0,1), (1,3,4), (2,4,0), (3,0,1), (4,1,2), (0,2,3), which constitute a cyclic ternary design with parameters $V=5, B=10,$

$$R=6,\ K=3,\ \pi_1=3,\ \pi_2=2,\ \triangle=8,\ n_1=n_2=2,\ p^1_{11}=0,\ p^2_{11}=1.$$

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